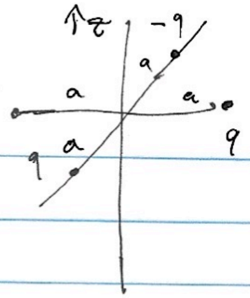
Jackson 4.1 (a).



$$q_{lm} = \int Y^*_{lm}(\theta', \phi')\, r'^l \rho(\vec{x}')\, d^3x.$$

~~$Y_{lm}(\theta,\phi) =$~~

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi).$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \Rightarrow q_{00} = 0.$$

$$\Rightarrow q_{lm} = Y^*_{lm}\left(\tfrac{\pi}{2}, 0\right) q\, a^l$$
$$+ Y^*_{lm}\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) q\, a^l$$
$$+ Y^*_{lm}\left(\tfrac{\pi}{2}, \pi\right) (-q)\, a^l$$
$$+ Y^*_{lm}\left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) (-q)\, a^l.$$

$$\Rightarrow q_{lm} = q a^l \left[Y^*_{lm}\left(\tfrac{\pi}{2}, 0\right) + Y^*_{lm}\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) - Y^*_{lm}\left(\tfrac{\pi}{2}, \pi\right) - Y^*_{lm}\left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) \right]$$

$\Rightarrow$ only ϕ-dependence ~~matter~~ can make q_{lm} vanish.

$\Rightarrow$ ~~m~~ $q_{l0} = 0.$

$m=1. \to e^{i0} + e^{i\pi/2}, -e^{i\pi}, -e^{i3\pi/2}$ ✓.

$m=2 \to \underbrace{e^{i0} + e^{i\pi}}_{0}, \underbrace{-e^{i2\pi}, -e^{i3\pi}}$. vanish,

0.

$m=3 \to e^{i0} + e^{i3\pi/2}, -e^{i3\pi}, -e^{i9\pi/2}.$

$e^{i0} + e^{i3\pi/2}, e^{i0}, e^{i3\pi/2}$ ✓.

$\Rightarrow$ $\boxed{\text{nonvanishing} \Rightarrow q_{11}, q_{1-1}, q_{33}, q_{3-3}.}$

Computing q_{11} explicitly : $Y_{11}^{*} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$.

$$q_{11} = e a^{1} \left(-\sqrt{\frac{3}{8\pi}}\right) \sin\frac{\pi}{2} \left[e^{0} + e^{-i\pi/2} - e^{-i\pi} - e^{-i3\pi/2} \right].$$

$$= -\sqrt{\frac{3}{8\pi}}\, e a \left[1 + 1 + (-i) - (i) \right].$$

$$= \boxed{-\sqrt{\frac{3}{8\pi}}\, 2 e a \left[1 - i \right].}$$

subsequently, $q_{1-1} = q_{11}^{*}(-1)$

$$= \boxed{\sqrt{\frac{3}{8\pi}}\, 2 e a (1 + i).}$$

We apply the same scheme to find q_{33}, q_{3-3}:

$Y_{33}^{*} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}} \sin^{3} e^{-3i\phi}$

$$q_{33} = e a^{3} \left(-\frac{1}{4}\right)\left(\sqrt{\frac{35}{4\pi}}\right) \sin^{3}\frac{\pi}{2} \times \left[e^{0} + e^{-3i\pi/2} - e^{-3i\pi} - e^{-9i\pi/2} \right].$$

$$= -\frac{ea^{3}}{4}\sqrt{\frac{35}{4\pi}} \times \left[1 + i - (-1) - (-i) \right].$$

$$= \frac{-2ea^{3}}{4}\sqrt{\frac{35}{4\pi}} \left[1 + i \right] = \boxed{-\frac{ea^{3}}{2}\sqrt{\frac{35}{4\pi}} \left[1 + i \right].}$$

Consequently, $q_{3-3} = \boxed{\frac{ea^{3}}{2}\sqrt{\frac{35}{4\pi}} \left[1 - i \right]}$

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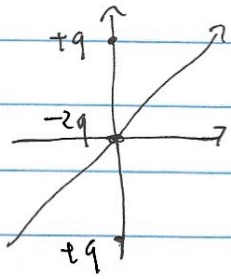
Jackson

9.1(b)



It's clear that this charge distribution is azimuthally symmetric, thus the expansion would be in legendre Poly. That is, the $m \neq 0$ terms are redundant. to consider.

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^{l} \rho(\vec{x}') d^3x'.$$

$$= Y^*_{lm}(0, 0) 1 (-2e) + \left[Y^*_{lm}(0, 0) a^l + Y^*_{lm}(\pi, 0) a^l \right] e.$$

$$= e\left[\left(Y^*_{lm}(0,0) + Y^*_{lm}(\pi, 0) \right) a^l - 2 Y^*_{lm}(0, 0) \right].$$

Evaluation at $\phi = 0$ eliminates all $m \neq 0$ terms.

$$q_{l0} = e\left[\left(Y^*_{l0}(0, 0) + Y^*_{l0}(\pi, 0) \right) a^l - 2 Y^*_{l0}(0, 0) \right].$$

$\Rightarrow q_{00} = 0.$

$\cancel{Y^*_{10} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, q_{40}} =$ For $m=0$, $Y_{l0} = P_l[\cos\theta] \sqrt{\frac{2l+1}{4\pi}}$.

$$\Rightarrow Y^*_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y^*_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right).$$

$$Y^*_{30} = \sqrt{\frac{7}{4\pi}} \left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta \right).$$

$$Y_{l0} = P_l[\cos\theta]\sqrt{\frac{2l+1}{4\pi}} \Rightarrow Y_{l0}^{*} = P_l[\cos\theta]\sqrt{\frac{2l+1}{4\pi}}.$$

$$\cos\big|_{\theta=0} = 1, \qquad \cos\pi = -1.$$

$$q_{l0} = e\sqrt{\frac{2l+1}{4\pi}}\left[(P_l[x=1]+P_l[x=-1])a^l - 2P_l[x=1]\right].$$

$$P_l[1] = 1, \qquad P_l[-1] = (-1)^l.$$

$$\Rightarrow q_{l0} = e\sqrt{\frac{2l+1}{4\pi}}\left[(1+(-1)^l)a^l - 2\right].$$

$$\boxed{q_{20} = e\sqrt{\frac{5}{4\pi}}\left[2a^l - 2\right].}$$

$$\boxed{q_{30} = e\sqrt{\frac{7}{4\pi}}\left[-2\right]}$$

$$\boxed{q_{l0} = e\sqrt{\frac{2l+1}{4\pi}}\left[(1+(-1)^l)a^l - 2\right]}$$

is the general form for nonvanishing terms.

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